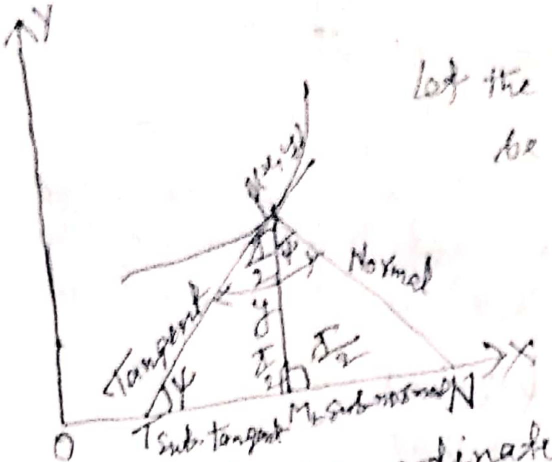
B.Sc. Part-II (H), Paper-III — 15-7-2020

Q.1. Prove that (i) the Cartesian sub-tangent $= \frac{y}{y_1}$
(ii) the Cartesian sub-normal $= yy_1$
(iii) the length of the tangent $= \frac{y}{y_1}\sqrt{1+y_1^2}$ and
(iv) the length of the normal $= y\sqrt{1+y_1^2}$ where $y_1 = \frac{dy}{dx}$

Soln:



Let the eqn of the curve be $y = f(x)$.
Let $P(x,y)$ be any point on it.

Let PM, PT, PN be the ordinate, tangent and normal to the curve and let PT make an angle ψ with the axis of x, then

$$\tan\psi = \frac{dy}{dx}$$

Let the tangent at P meet the x-axis in T. Then, the length TM is called the sub-tangent at P and the length MN is called the sub-normal at P.

In $\triangle$ PTM and PNM [Since both are right angled triangles]

Since $\angle MPN = \angle PTN = \psi$ and $PM = y$.

(I) Sub-tangent $= TM = y\cot\psi = \frac{y}{\tan\psi} = \frac{y}{\frac{dy}{dx}}$

(II) Sub-normal $= MN = y\tan\psi = y\frac{dy}{dx}$

(III) Tangent $= PT = y\,\mathrm{cosec}\,\psi = y\sqrt{1+\cot^2\psi}$

$$= \frac{y\sqrt{1+\tan^2\psi}}{\tan\psi} = y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$$

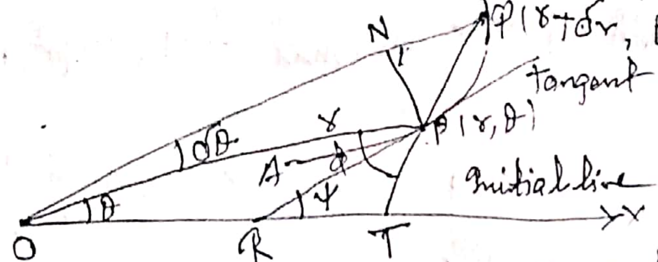
(IV) Normal $PN = y\sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Proved.

Q. Prove that (I) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ 15-07-2020

(II) $\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$ (III) $\tan\phi = r\frac{d\theta}{dr}$.

Soln:



Let O be the pole and OX the initial line.

Let A be the fixed point on the curve and P be any point (r,θ) on it such that arc $AP = s$.

Let $Q(r+\delta r, \theta+\delta\theta)$ be another point very close to P so that

$$\text{arc } AQ = s + \delta s$$

$$\therefore \text{arc } PQ = \delta s$$

Let PN be the perpendicular from P to OQ.

(I) From right angled triangle PNQ.

$$PQ^2 = PN^2 + NQ^2 = (r\sin\delta\theta)^2 + (OQ - ON)^2$$

$$= (r\sin\delta\theta)^2 + (r+\delta r - r\cos\delta\theta)^2$$

$$= (r\sin\delta\theta)^2 + [r(1-\cos\delta\theta) + \delta r]^2$$

$$= (r\sin\delta\theta)^2 + \left(r \cdot 2\sin^2\frac{\delta\theta}{2} + \delta r\right)^2$$

Dividing by $(\delta\theta)^2$, we get

$$\left(\frac{PQ}{\delta s}\cdot\frac{\delta s}{\delta\theta}\right)^2 = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left[\frac{1}{2}r\delta\theta\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 + \frac{\delta r}{\delta\theta}\right]^2$$

Let $Q \to P$, so that $\delta\theta \to 0$

$$\therefore 1\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

(II) multiplying both sides of (I) by $\frac{d\theta}{dr}$ we get

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$$

(III) $\because PN = r\sin\delta\theta$ and $QN = OQ - ON = r + \delta r - r\cos\delta\theta$

$$= r(1-\cos\delta\theta) + \delta r = \delta r + 2r\sin^2\frac{\delta\theta}{2}$$

$$\tan PQN = \frac{PN}{QN} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\frac{\delta\theta}{2}} = \frac{r\frac{\sin\delta\theta}{\delta\theta}}{\frac{\delta r}{\delta\theta} + \frac{1}{2}r\delta\theta\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2}$$

Let $Q \to P$, so that $\delta\theta \to 0$ and $\angle PQN \to \angle TPO = \phi$

Hence, $\tan\phi = \frac{r}{\frac{dr}{d\theta}} = r\frac{d\theta}{dr}$ Proved.

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